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Dimensional reduction of nonlinear gauge theories

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Abstract: We extend 2D nonlinear gauge theory from the Poisson sigma model based on Lie algebroid to a model with additional two-form gauge fields. Dimensional reduction of 3D nonlinear gauge theory yields an example of such a model, which provides a realization of Courant algebroid by 2D nonlinear gauge theory. We see that the reduction of the base structure generically results in a modification of the target (algebroid) structure.

Keywords: Field Theories in Lower Dimensions, Topological Field Theories, Gauge Symmetry, BRST Symmetry.

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Contents

1. Introduction

Topological gauge field theories of the Schwarz type [1] are grouped into two categories: Chern-Simons-BF gauge theory [1] and nonlinear gauge theory [2, 3]. Topological nonlinear gauge theory can be obtained as deformations of the former [4] and may be regarded as the most general form of the topological gauge field theories of the Schwarz type.

Two-dimensional nonlinear gauge theory of the simplest form [3] turns out to be determined solely by the Lie algebroid structure [5], the structure functions of nonlinear Lie (finite W [6]) algebra, or the data of Poisson algebra [7], which is manifested in the name of the Poisson sigma model [8].[1]

Various extensions of the 2D nonlinear gauge theory may be considered. In fact, 2D theory based on graded structures was obtained [9] soon after the exposition of the original bosonic model. Higher-dimensional generalization of the 2D nonlinear gauge theory is also possible [4]. Accordingly, 3D nonlinear gauge theory is systematically constructed [10]–[13][2] through deformations of Chern-Simons-BF gauge theories and the Courant algebroid [14] structure underlying it is identified [12, 13].

In this paper, we extend 2D nonlinear gauge theory from the Poisson sigma model based on the Lie algebroid to a model with additional two-form gauge fields. This extension is motivated by considering dimensional reduction of 3D nonlinear gauge theory based on the Courant algebroid.

[1]Since it has scalar fields in its field contents, it is apparently a sigma model as well as a gauge theory.
[2]Yet higher-dimensional cases are also considered in ref. [4, 11, 13].

2. 2D nonlinear gauge theory with two-forms

Let us first present an extended action of 2D nonlinear gauge theory postponing its derivation to the following sections.

We consider an action with 2-forms $\tilde{B}_{2A}$,

$$S = \int_\Sigma h_A d\Phi^A + \frac{1}{2} W^{AB}(\Phi) h_A h_B + V^A(\Phi) \tilde{B}_{2A} \,, \tag{2.1}$$

where Σ denotes a two-dimensional base manifold and M a target space manifold of a smooth map $\Phi : \Sigma \to M$ with a local coordinate expression $\{\Phi^A\}$. Here h_A is a section of $T^*\Sigma \otimes \Phi^*(TM)$, $\tilde{B}_{2A}$ is a section of $\wedge^2 T^*\Sigma \otimes \Phi^*(TM)$, and $W^{AB} = -W^{BA}$ denotes a bivector field and V^A a vector field on the target space M. When $V^A(\Phi) \equiv 0$, the action (2.1) reduces to the Poisson sigma model.

The action has the following gauge symmetry:

$$\begin{aligned}
\delta\Phi^A &= -W^{AB} c_B \,, \\
\delta h_A &= dc_A + \frac{\partial W^{BC}}{\partial \Phi^A} h_B c_C - \frac{\partial V^B}{\partial \Phi^A} t_B \,, \\
\delta \tilde{B}_{2A} &= dt_A + U_A{}^{BC}(h_B t_C - \tilde{B}_{2C} c_B) + \frac{1}{2} X_A{}^{BCD} h_B h_C c_D \,,
\end{aligned} \tag{2.2}$$

if $U_A{}^{BC}(\Phi)$ and $X_A{}^{BCD}(\Phi)$ satisfy the identities

$$\begin{aligned}
W^{D[A} \frac{\partial W^{BC]}}{\partial \Phi^D} &= V^D X_D{}^{ABC} \,, \\
W^{AB} \frac{\partial V^C}{\partial \Phi^A} + U_A{}^{BC} V^A &= 0 \,,
\end{aligned} \tag{2.3}$$

where c_A is a 0-form gauge parameter and t_A is a 1-form gauge parameter with $X_A{}^{BCD}$ completely antisymmetric with respect to the indices BCD.

In fact, the action S is gauge invariant

$$\delta S = \int_\Sigma d(c_A d\Phi_A + V^A t_A) \,, \tag{2.4}$$

and its equations of motion are given by

$$\begin{aligned}
dh_A + \frac{1}{2} \frac{\partial W^{BC}}{\partial \Phi^A} h_B h_C + \frac{\partial V^B}{\partial \Phi^A} \tilde{B}_{2B} &= 0 \,, \\
d\Phi^A + W^{AB} h_B &= 0 \,, \\
V^A &= 0 \,.
\end{aligned} \tag{2.5}$$

If we can take $V^A = (0, V^a(\Phi^A))$ for $\{a\} \subset \{A\}$ with $V^a(\Phi^A) = \Phi^a$ through a coordinate transformation on M,[3] we may locally eliminate the fields Φ^a by means of the equations of motion (let us call the extended theory *reducible* in this case). Then the theory reduces to the Poisson sigma model with the target space dimension reduced accordingly.

[3] This is possible locally for generic $V^a(\Phi^A)$.

3. Derivation of the 2D theory with two-forms

In this section, we derive the action proposed in the previous section through a deformation of BF theory in two dimensions. We first set up a superformalism [15, 16] of the BF theory and then perform a consistent deformation [17] thereof.

3.1 Superformalism of 2D BF theory with two-forms

We start with a free action

$$S_{\rm A} = \int_\Sigma h_A d\Phi^A \,, \tag{3.1}$$

where h_A is a 1-form gauge field and Φ^A is a 0-form scalar field on Σ. We further introduce $\tilde{B}_{2A}$ as a 2-form field on Σ. This action has an abelian gauge symmetry

$$\begin{aligned} \delta_0 h_A &= dc_A \,, \\ \delta_0 \tilde{B}_{2A} &= dt_A \,, \end{aligned} \tag{3.2}$$

where c_A is a 0-form gauge parameter and t_A is a 1-form gauge parameter. The gauge symmetry for $\tilde{B}_{2A}$ is trivially satisfied and reducible. We need the following tower of the 'ghosts for ghosts' to analyze the complete gauge degrees of freedom:

$$\begin{aligned} \delta_0 h_A &= dc_A \,, \qquad \delta_0 c_A = 0 \,, \\ \delta_0 \tilde{B}_{2A} &= dt_A \,, \qquad \delta_0 t_A = dv_A \,, \qquad \delta_0 v_A = 0 \,, \end{aligned} \tag{3.3}$$

where v_A is a 0-form gauge parameter.

Let us here set up the Batalin-Vilkovisky formalism [15] to adopt the Barnich-Henneaux approach for consistent deformation [17] in the next subsection. First we take c_A to be the FP ghost 0-form with the ghost number 1, t_A to be a 1-form with the ghost number 1, and v_A to be a 0-form with the ghost number 2. Next we introduce the antifields for all the fields: φ^+ denotes the antifield for the field φ. Note that such relations as $\mathrm{gh}(\varphi) + \mathrm{gh}(\varphi^+) = -1$ and $\deg(\varphi) + \deg(\varphi^+) = 2$ (in two dimensions) are imposed, where $\mathrm{gh}(\varphi)$ and $\mathrm{gh}(\varphi^+)$ are the ghost numbers of the (anti)fields φ and φ^+ and $\deg(\varphi)$ and $\deg(\varphi^+)$ are their form degrees, respectively.

In order to simplify combinatorics, we adopt a superformalism [16]. Namely, we combine the fields, antifields, and their gauge descendant fields as superfield components:

$$\begin{aligned} \boldsymbol{h}_A &= c_A + h_A + \Phi^+_A \,, \\ \boldsymbol{\Phi}^A &= \Phi^A + h^{+A} + c^{+A} \,, \\ \boldsymbol{B}_A &= v_A + t_A + \tilde{B}_{2A} \,, \\ \boldsymbol{B}^{+A} &= \tilde{B}_2{}^{+A} + t^{+A} + v^{+A} \,. \end{aligned} \tag{3.4}$$

Note that the component fields F in a superfield have the common *total degree* $|F| \equiv \mathrm{gh} F + \deg F$. The total degrees of $\boldsymbol{h}_A$, $\boldsymbol{\Phi}^A$, $\boldsymbol{B}_A$, and $\boldsymbol{B}^{+A}$ are 1, 0, 2, and -1, respectively.

We introduce the *antighost number* antigh($\boldsymbol{F}$) of a superfield $\boldsymbol{F}$ with only antigh($\boldsymbol{B}^{+A}$) $= 1$ nonvanishing. The super antibracket of two superfields $\boldsymbol{F}$ and $\boldsymbol{G}$ is given by

$$
\begin{aligned}
(\boldsymbol{F},\, \boldsymbol{G}) = \boldsymbol{F} \cdot \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\Phi}^A} \cdot \frac{\overrightarrow{\partial}}{\partial \boldsymbol{h}_A} \cdot \boldsymbol{G} - \boldsymbol{F} \cdot \frac{\overleftarrow{\partial}}{\partial \boldsymbol{h}_A} \cdot \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\Phi}^A} \cdot \boldsymbol{G} + \\
+ \boldsymbol{F} \cdot \frac{\overleftarrow{\partial}}{\partial \boldsymbol{B}_A} \cdot \frac{\overrightarrow{\partial}}{\partial \boldsymbol{B}^{+A}} \cdot \boldsymbol{G} - \boldsymbol{F} \cdot \frac{\overleftarrow{\partial}}{\partial \boldsymbol{B}^{+A}} \cdot \frac{\overrightarrow{\partial}}{\partial \boldsymbol{B}_A} \cdot \boldsymbol{G}\,,
\end{aligned}
\tag{3.5}
$$

where we have utilized the *super product*, the *super antibracket*, and the *super differentiation* defined in the appendix.

We can now construct the Batalin-Vilkovisky action to the original action (3.1) with the superfields as follows:

$$
\begin{aligned}
S_0 &= \int_\Sigma \boldsymbol{h}_A \cdot d\boldsymbol{\Phi}^A + \boldsymbol{B}^{+A} \cdot d\boldsymbol{B}_A\,, \\
&= \int_\Sigma h_A d\Phi^A - c_A dh^{+A} - \tilde{B}_2{}^{+A} dt_A + t^{+A} dv_A\,,
\end{aligned}
\tag{3.6}
$$

where only the 2-form part of the integrand survives integration on the two-dimensional manifold Σ. The total degree of the integrand of S_0 is two and its antighost number is zero. If we set all the antifields equal to zero, eq. (3.6) reduces to eq. (3.1).

If we assigned the total degrees of superfields $\boldsymbol{h}_A$, $\boldsymbol{\Phi}^A$, $\boldsymbol{B}_A$, and $\boldsymbol{B}^{+A}$ as 1, 0, 0, and 1, respectively, we could regard the superfield action (3.6) as a Batalin-Vilkovisky action to the usual abelian BF theory. In fact, there is a one-to-one map in terms of superfield actions from the 2D abelian BF theory with two-forms to the 2D abelian BF theory, which changes the total degrees of superfields $\boldsymbol{B}_A$ and $\boldsymbol{B}^{+A}$ from 2 and -1 to 0 and 1, respectively.

The BRST transformation of the superfield $\boldsymbol{F}$ for the above action is given by

$$
\delta_0 \boldsymbol{F} \equiv (S_0,\, \boldsymbol{F})\,,
\tag{3.7}
$$

which yields

$$
\begin{aligned}
\delta_0 \boldsymbol{h}_A &= (S_0,\, \boldsymbol{h}_A) = d\boldsymbol{h}_A\,, \\
\delta_0 \boldsymbol{\Phi}^A &= (S_0,\, \boldsymbol{\Phi}^A) = d\boldsymbol{\Phi}^A\,, \\
\delta_0 \boldsymbol{B}_A &= (S_0,\, \boldsymbol{B}_A) = d\boldsymbol{B}_A\,, \\
\delta_0 \boldsymbol{B}^{+A} &= (S_0,\, \boldsymbol{B}^{+A}) = d\boldsymbol{B}^{+A}\,.
\end{aligned}
\tag{3.8}
$$

By expanding these BRST transformations to the components eq. (3.4), we obtain the BRST transformation of each field and antifield as follows:

$$
\begin{aligned}
\delta_0 \Phi^+_A &= dh_A\,, & \delta_0 h_A &= dc_A\,, & \delta_0 c_A &= 0\,, \\
\delta_0 c^{+A} &= dh^{+A}\,, & \delta_0 h^{+A} &= d\Phi^A\,, & \delta_0 \Phi^A &= 0\,, \\
\delta_0 \tilde{B}_{2A} &= dt_A\,, & \delta_0 t_A &= dv_A\,, & \delta_0 v_A &= 0\,, \\
\delta_0 v^{+A} &= dt^{+A}\,, & \delta_0 t^{+A} &= d\tilde{B}_2{}^{+A}\,, & \delta_0 \tilde{B}_2{}^{+A} &= 0\,,
\end{aligned}
\tag{3.9}
$$

which reproduces the original gauge transformation (3.3). It is simple to confirm that S_0 is BRST invariant and $\delta_0^2 = 0$ on all the fields.

3.2 Consistent deformation of the Batalin-Vilkovisky action

Let us consider a deformation of the action S_0,

$$S = S_0 + gS_1 + g^2 S_2 + \cdots , \tag{3.10}$$

where g is a deformation parameter or a coupling constant of the theory.

For a consistent deformation [17], we demand the total action S to satisfy the classical master equation

$$(S, S) = 0 . \tag{3.11}$$

Substituting eq. (3.10) to eq. (3.11), we obtain the g power expansion of the master equation

$$(S, S) = (S_0, S_0) + 2g(S_0, S_1) + g^2[(S_1, S_1) + 2(S_0, S_2)] + \mathcal{O}(g^3) = 0 . \tag{3.12}$$

We solve this equation order by order with physical requirements for the solutions: We require the Lorentz invariance (lorentzian case) or SO(2) invariance (euclidean case) of the action. We assume that S is *local*, that is, S is given by the integration of a local lagrangian: $S = \int_\Sigma \mathcal{L}$. Note that we exclude the solutions whose BRST transformations are not deformed ($\delta = \delta_0$) as trivial ones. This condition is implied by the assumption that each term contains at least one antifield in S_i for $i \geq 1$.

At the 0-th order, we obtain $\delta_0 S_0 = (S_0, S_0) = 0$, which is already satisfied. At the first order of g in eq. (3.12),

$$\delta_0 S_1 = (S_0, S_1) = 0 \tag{3.13}$$

is required. S_1 is given by the integration of a local lagrangian from the assumption:

$$S_1 = \int_\Sigma \mathcal{L}_1 , \tag{3.14}$$

where $\mathcal{L}_1$ can be constructed from the superfields $\boldsymbol{h}_A$, $\boldsymbol{\Phi}^A$, $\boldsymbol{B}_A$, and $\boldsymbol{B}^{+A}$. If a monomial in $\mathcal{L}_1$ includes a differential d, it is proportional to the free equations of motion. Therefore it can be absorbed into the abelian action (3.6) through local field redefinitions of superfields and these terms are BRST trivial in the BRST cohomology. Hence the nontrivial deformation does not include the differential d, and thus $\mathcal{L}_1$ is a degree two function of the superfields $\boldsymbol{h}_A$, $\boldsymbol{\Phi}^A$, $\boldsymbol{B}_A$, and $\boldsymbol{B}^{+A}$.

At the second order of g,

$$(S_1, S_1) + 2(S_0, S_2) = 0 \tag{3.15}$$

is required. We cannot construct nontrivial S_2 to satisfy eq. (3.15) from the integration of a local lagrangian, since δ_0-BRST transforms of the local terms are always total derivative. Therefore, if we assume locality of the action, S_2 is BRST trivial (the Poincaré lemma), that is, we obtain the relation $(S_0, S_2) = 0$ and we can absorb S_2 into S_1. Similarly, when we solve the higher order g part of eq. (3.12) recursively, we find that S_i is BRST trivial for $i \geq 2$. Hence we may set $S_i = 0$ for $i \geq 2$. Then the condition (3.11) reduces to

$$(S_1, S_1) = 0 \,. \tag{3.16}$$

This equation imposes conditions on the structure functions $f_i(\mathbf{\Phi})$ in eq. (3.18).

Let us solve the condition (3.16) explicitly. We expand S_1 by the antighost number

$$S_1 = \sum_k S_1^{(k)} = \sum_k \int_\Sigma \mathcal{L}_1^{(k)} \,, \tag{3.17}$$

where $S_1^{(k)}$ is the antighost number k part of the action S_1 and $\mathcal{L}_1^{(k)}$ is its lagrangian ($k \geq 0$). We can write the candidate $\mathcal{L}_1^{(k)}$ under the requirement $|\mathcal{L}_1^{(k)}| = 2$ as

$$\begin{aligned} \mathcal{L}_1^{(0)} &= \frac{1}{2} f_1^{AB}(\mathbf{\Phi}) \cdot \boldsymbol{h}_A \cdot \boldsymbol{h}_B + f_2^A(\mathbf{\Phi}) \cdot \boldsymbol{B}_A \,, \\ \mathcal{L}_1^{(1)} &= \frac{1}{3!} f_{3A}{}^{BCD}(\mathbf{\Phi}) \cdot \boldsymbol{B}^{+A} \cdot \boldsymbol{h}_B \cdot \boldsymbol{h}_C \cdot \boldsymbol{h}_D + f_{4A}{}^{BC}(\mathbf{\Phi}) \cdot \boldsymbol{B}^{+A} \cdot \boldsymbol{h}_B \cdot \boldsymbol{B}_C \,, \end{aligned} \tag{3.18}$$

and so on, where $f_i(\mathbf{\Phi})$ is a function of $\mathbf{\Phi}^A$.

We also expand (S_1, S_1) by the antighost number as follows:

$$(S_1, S_1) = \sum_k (S_1, S_1)^{(k)} = 0 \,, \tag{3.19}$$

where $(S_1, S_1)^{(k)}$ is the antighost number k part of (S_1, S_1). This equation requires $(S_1, S_1)^{(k)} = 0$ for all the nonnegative integers k, and we can determine the conditions among f_i recursively: First, we consider $k = 0$ part. We substitute eq. (3.18) to $(S_1, S_1)^{(0)} = 0$ and obtain

$$\begin{aligned} f_1^{D[A} \frac{\partial f_1^{BC]}}{\partial \Phi^D} + f_2^D f_{3D}{}^{ABC} &= 0 \,, \\ - f_1^{AB} \frac{\partial f_2^C}{\partial \Phi^A} + f_{4A}{}^{BC} f_2^A &= 0 \,. \end{aligned} \tag{3.20}$$

As for the higher order terms with respect to the antighost number, we obtain conditions of higher order f_i's. It is sufficient to consider $k \leq 1$ terms in order to obtain the deformed action and deformed BRST transformation, since higher order terms vanish when we set all the antifields equal to zero.

The BRST transformation of each superfield is given by

$$\begin{aligned} \delta \boldsymbol{h}_A &= d\boldsymbol{h}_A + \frac{1}{2} g \frac{\partial f_1^{BC}}{\partial \mathbf{\Phi}^A} \cdot \boldsymbol{h}_B \cdot \boldsymbol{h}_C + g \frac{\partial f_2^B}{\partial \mathbf{\Phi}^A} \cdot \boldsymbol{B}_B + \frac{1}{3!} g \frac{\partial f_{3B}{}^{CDE}}{\partial \mathbf{\Phi}^A} \cdot \boldsymbol{B}^{+B} \cdot \boldsymbol{h}_C \cdot \boldsymbol{h}_D \cdot \boldsymbol{h}_E + \\ &\quad + \frac{1}{2} g \frac{\partial f_{4B}{}^{CD}}{\partial \mathbf{\Phi}^A} \cdot \boldsymbol{B}^{+B} \cdot \boldsymbol{h}_C \cdot \boldsymbol{B}_D + \cdots \,, \\ \delta \mathbf{\Phi}^A &= d\mathbf{\Phi}^A - g f_1^{AB} \cdot \boldsymbol{h}_B + \frac{1}{2} g f_{3B}{}^{ACD} \cdot \boldsymbol{B}^{+B} \cdot \boldsymbol{h}_C \cdot \boldsymbol{h}_D - g f_{4B}{}^{AC} \cdot \boldsymbol{B}^{+B} \cdot \boldsymbol{B}_C + \cdots \,, \\ \delta \boldsymbol{B}_A &= d\boldsymbol{B}_A - \frac{1}{3!} g f_{3A}{}^{BCD} \cdot \boldsymbol{h}_B \cdot \boldsymbol{h}_C \cdot \boldsymbol{h}_D - g f_{4A}{}^{BC} \cdot \boldsymbol{h}_B \cdot \boldsymbol{B}_C + \cdots \,, \\ \delta \boldsymbol{B}^{+A} &= d\boldsymbol{B}^{+A} - g f_2^A - g f_{4B}{}^{CA} \cdot \boldsymbol{B}^{+B} \cdot \boldsymbol{h}_C + \cdots \,, \end{aligned} \tag{3.21}$$

where the ellipses represent the terms stemming from higher $S_1^{(k)}$ ($k \geq 2$).

For all the antifields vanishing, we finally arrive at the action (2.1) with gauge symmetry (2.2) in the previous section. Accordingly, if we set

$$
\begin{aligned}
g f_1^{AB} &= W^{AB}, \qquad g f_2^{A} = V^{A}, \\
g f_{3A}{}^{BCD} &= -X_A{}^{BCD}, \qquad g f_{4A}{}^{BC} = -U_A{}^{BC},
\end{aligned}
\tag{3.22}
$$

eq. (3.20) coincides with eq. (2.3).

4. Dimensional reduction of 3D theories

In this section, we dimensionally reduce the 3D nonlinear gauge theory based on Courant algebroid to 2D theory.

Let X be a three-dimensional manifold with a coordinate (τ, σ, ρ) and M be a target manifold of a smooth map $\phi : X \to M$ with local coordinate expression $\{\phi^i\}$. We also have a vector bundle E over M with A^a a section of $T^*X \otimes \phi^*(E^*)$.

We can construct 3D topological gauge field theory of the Schwarz type in terms of ϕ^i and A^a [4, 10, 11, 12, 13]. For that purpose, we further introduce B_{1a} as a section of $T^*X \otimes \phi^*(E)$ and B_{2i} as a section of $\wedge^2 T^*X \otimes \phi^*(TM)$. Hereafter, the letters $a, b, \cdots$ represent indices on the fiber of E and $i, j, \cdots$ represent indices on M and TM.

4.1 3D theory based on Lie algebroid

As a simplest example, let us first try 3D nonlinear BF theory with nonlinear gauge symmetry based on Lie algebroid or in the case with the target M a Poisson manifold equipped with a Poisson bivector $\omega^{ij}(\phi) = -\omega^{ji}(\phi)$ and $E = TM$.

A Lie algebroid over a manifold M is a vector bundle $E \to M$ with a Lie algebra structure on the space of the sections $\Gamma(E)$ defined by the bracket $[e_1, e_2]$ for $e_1, e_2 \in \Gamma(E)$ and a bundle map (the anchor) $\rho : E \to TM$ satisfying the following properties:

$$
\begin{aligned}
&\text{for any} \quad e_1, e_2 \in \Gamma(E), \qquad [\rho(e_1), \rho(e_2)] = \rho([e_1, e_2]); \\
&\text{for any} \quad e_1, e_2 \in \Gamma(E), \quad F \in C^\infty(M), \qquad [e_1, F e_2] = F[e_1, e_2] + (\rho(e_1)F)e_2 .
\end{aligned}
\tag{4.1}
$$

If $E = TM$ and M is a Poisson manifold, a Poisson bivector $\omega(\phi)$ defines a Lie algebroid: We take $\{e^i\}$ as a local basis of $\Gamma(TM)$ and let a local expression of a Poisson bivector be $\omega^{ij}(\phi) = -\omega^{ji}(\phi)$. Then we can define a Lie algebroid by the following equations:

$$
\begin{aligned}
[e^i, e^j] &= \frac{\partial \omega^{ij}(\phi)}{\partial \phi^k} e^k \\
\rho(e^i) &= \omega^{ij}(\phi) \frac{\partial}{\partial \phi^j} .
\end{aligned}
\tag{4.2}
$$

We now adopt 3D nonlinear gauge theory with an action [4, 10]

$$
\begin{aligned}
S &= S_0 + S_1 ; \\
S_0 &= \int_X \left(B_{1i} \wedge dA^i - B_{2i} \wedge d\phi^i \right), \\
S_1 &= \int_X \left(\omega^{ij}(\phi) B_{2i} B_{1j} + \frac{1}{2} \omega_i^{jk}(\phi) A^i B_{1j} B_{1k} \right),
\end{aligned}
\tag{4.3}
$$

where we have defined

$$\omega_i^{jk}(\phi) \equiv \frac{\partial \omega^{jk}(\phi)}{\partial \phi^i} . \tag{4.4}$$

We consider dimensional reduction of the theory from the three-dimensional manifold $X = \Sigma \times S^1$ to the two-dimensional manifold Σ. Namely, all the fields are set independent of the coordinate ρ of S^1 with $\int_{S^1} d\rho = 1$:

$$\begin{aligned} \phi^i &= \tilde{\phi}^i , \\ A^i &= \tilde{A}_1{}^i + \tilde{\alpha}_0{}^i d\rho , \\ B_{1i} &= \tilde{B}_{1i} + \tilde{\beta}_{0i} d\rho , \\ B_{2i} &= \tilde{B}_{2i} + \tilde{\beta}_{1i} d\rho , \end{aligned} \tag{4.5}$$

where $\tilde{\phi}^i$ is a reduction of ϕ^i, $\tilde{A}_1{}^i$ and $\tilde{B}_{1i}$ are 1-forms, $\tilde{\alpha}_0{}^i$ and $\tilde{\beta}_{0i}$ are 0-forms, $\tilde{B}_{2i}$ is a 2-form, and $\tilde{\beta}_{1i}$ is a 1-form in two dimensions.

Then the action (4.3) is reduced to the following action:

$$\begin{aligned} S = \int_X & \left(\tilde{B}_{1i} \wedge d\tilde{\alpha}_0{}^i + \tilde{A}_1{}^i \wedge d\tilde{\beta}_{0i} + \tilde{\beta}_{1i} \wedge d\tilde{\phi}^i\right) d\rho + d\left(\tilde{A}_1{}^i \tilde{\beta}_{0i} d\rho\right) + \\ & + \left(\omega^{ij}(\tilde{\phi})(\tilde{B}_{2i}\tilde{\beta}_{0j} - \tilde{\beta}_{1i}\tilde{B}_{1j}) + \frac{1}{2}\omega_i^{jk}(\tilde{\phi})(2\tilde{A}_1{}^i \tilde{B}_{1j}\tilde{\beta}_{0k} + \tilde{\alpha}_0{}^i \tilde{B}_{1j}\tilde{B}_{1k})\right) d\rho , \end{aligned} \tag{4.6}$$

which can be also obtained through the action (2.1) by letting

$$\begin{aligned} h_A &= (\tilde{\beta}_{1i}, \tilde{A}_1{}^j, \tilde{B}_{1k}) , \\ \Phi^A &= (\tilde{\phi}^i, \tilde{\beta}_{0j}, \tilde{\alpha}_0{}^k) , \\ \tilde{B}_{2A} &= (\tilde{B}_{2i}, 0, 0) ; \\ W^{AB} &= \begin{pmatrix} 0 & 0 & -\omega^{\mathrm{in}} \\ 0 & 0 & \omega_j^{np}\tilde{\beta}_{0p} \\ \omega^{lk} & -\omega_k^{mp}\tilde{\beta}_{0p} & \omega_p^{kn}\tilde{\alpha}_0{}^p \end{pmatrix} , \\ V^A &= (\omega^{ip}\tilde{\beta}_{0p}, 0, 0) , \end{aligned} \tag{4.7}$$

for ${}^A = ({}^i, {}_j, {}^k)$.

The 3D nonlinear gauge theory based on Lie algebroid from X to M reduces to a 2D nonlinear gauge theory with two-forms from Σ to $TM \oplus T^*M$ as a sigma model by dimensional reduction.

When the manifold M is symplectic or the ω^{ij} is invertible, we may eliminate the fields $\tilde{\beta}_{0j}$ and $\tilde{B}_{2i}$ by means of the equations of motion. Then we further obtain a Poisson sigma model as the reduced theory (that is, the 2D theory above is reducible) with

$$\begin{aligned} h_A &= (\tilde{\beta}_{1i}, \tilde{B}_{1k}) , \\ \Phi^A &= (\tilde{\phi}^i, \tilde{\alpha}_0{}^k) , \\ W^{AB} &= \begin{pmatrix} 0 & -\omega^{in} \\ \omega^{lk} & \omega_p^{kn}\tilde{\alpha}_0{}^p \end{pmatrix} , \end{aligned} \tag{4.8}$$

for $A = (i, k)$.

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4.2 Nonlinear Chern-Simons theory

We can generally construct nonlinear Chern-Simons theory with nonlinear gauge symmetry as a deformation of the Chern-Simons gauge theory. This nonlinear gauge theory in three dimensions has the following action [12]:

$$\begin{aligned} S &= S_0 + S_1 \,, \\ S_0 &= \int_X \left(\frac{k_{ab}}{2} A^a \wedge dA^b - B_{2i} \wedge d\phi^i \right), \\ S_1 &= \int_X \left(f_{1a}{}^i(\phi) A^a B_{2i} + \frac{1}{6} f_{2abc}(\phi) A^a A^b A^c \right), \end{aligned} \tag{4.9}$$

where k_{ab} is a symmetric constant tensor and the structure functions f_1 and f_2 satisfy the identities

$$\begin{aligned} k^{ab} f_{1a}{}^i f_{1b}{}^j &= 0 \,, \\ \frac{\partial f_{1b}{}^i}{\partial \phi^j} f_{1c}{}^j - \frac{\partial f_{1c}{}^i}{\partial \phi^j} f_{1b}{}^j + k^{ef} f_{1e}{}^i f_{2fbc} &= 0 \,, \\ \left(f_{1d}{}^j \frac{\partial f_{2abc}}{\partial \phi^j} - f_{1c}{}^j \frac{\partial f_{2dab}}{\partial \phi^j} + f_{1b}{}^j \frac{\partial f_{2cda}}{\partial \phi^j} - f_{1a}{}^j \frac{\partial f_{2bcd}}{\partial \phi^j} \right) + & \\ + k^{ef} (f_{2eab} f_{2cdf} + f_{2eac} f_{2dbf} + f_{2ead} f_{2bcf}) &= 0 \,. \end{aligned} \tag{4.10}$$

We assume that k_{ab} is nondegenerate or invertible.

A Courant algebroid [14] is a vector bundle $E \to M$ with a nondegenerate symmetric bilinear form $\langle \cdot , \cdot \rangle$ on the bundle, a bilinear operation $\circ$ on $\Gamma(E)$ (the space of sections on E), and a bundle map (called the anchor) $\rho : E \to TM$ satisfying the following properties:

$$\begin{aligned} &1) \quad e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) \,, \\ &2) \quad \rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)] \,, \\ &3) \quad e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1) F) e_2 \,, \\ &4) \quad e_1 \circ e_2 = \frac{1}{2} \mathcal{D} \langle e_1 \,, e_2 \rangle \,, \\ &5) \quad \rho(e_1) \langle e_2 \,, e_3 \rangle = \langle e_1 \circ e_2 \,, e_3 \rangle + \langle e_2 \,, e_1 \circ e_3 \rangle \,, \end{aligned} \tag{4.11}$$

where e_1, e_2, and e_3 are sections of E; F is a function on M; $\mathcal{D}$ is a map from functions on M to $\Gamma(E)$ and is defined by $\langle \mathcal{D}F \,, e \rangle = \rho(e) F$.

If we take a local basis, eq. (4.10) is equivalent to the relations 1) to 5) of structure functions of a Courant algebroid on a vector bundle E over M: We take a basis e^a of $\Gamma(E)$. Then symmetric bilinear form is defined by $\langle e^a \,, e^b \rangle = k^{ab}$. The bilinear operation is defined by $e^a \circ e^b = -k^{ac} k^{bd} f_{2cde}(\phi) e^e$ and the anchor is defined by $\rho(e^a) = -f_{1c}{}^i(\phi) k^{ac} \frac{\partial}{\partial \phi^i}$.

We again consider dimensional reduction of the theory from the three-dimensional manifold $X = \Sigma \times S^1$ to the two-dimensional manifold Σ:

$$\begin{aligned} \phi^i &= \tilde{\phi}^i \,, \\ A^a &= \tilde{A}_1{}^a + \tilde{\alpha}_0{}^a d\rho \,, \\ B_{2i} &= \tilde{B}_{2i} + \tilde{\beta}_{1i} d\rho \,, \end{aligned} \tag{4.12}$$

where $\tilde{\phi}^i$ is a reduction of ϕ^i, $\tilde{A}_1{}^a$ is a 1-form, $\tilde{\alpha}_0{}^a$ is a 0-form, $\tilde{B}_{2i}$ is a 2-form, and $\tilde{\beta}_{1i}$ is a 1-form in two dimensions.

Then the action (4.9) is reduced to the following action:

$$\begin{aligned} S = \int_X & \left(k_{ab}\tilde{A}_1{}^a \wedge d\tilde{\alpha}_0{}^b + \tilde{\beta}_{1i} \wedge d\tilde{\phi}^i\right) d\rho + d\left(\frac{k_{ab}}{2}\tilde{A}_1{}^a\tilde{\alpha}_0{}^b d\rho\right) + \\ & + \left(f_{1a}{}^i(\tilde{\phi})(\tilde{A}_1{}^a\tilde{\beta}_{1i} + \tilde{\alpha}_0{}^a\tilde{B}_{2i})\frac{1}{2}f_{2abc}(\tilde{\phi})\tilde{A}_1{}^a\tilde{A}_1{}^b\tilde{\alpha}_0{}^c\right) d\rho\,, \end{aligned} \tag{4.13}$$

which can be also obtained through the action (2.1) by letting

$$\begin{aligned} h_A &= (\tilde{\beta}_{1i}, k_{ab}\tilde{A}_1{}^b)\,, \\ \Phi^A &= (\tilde{\phi}^i, \tilde{\alpha}_0{}^a)\,, \\ \tilde{B}_{2A} &= (\tilde{B}_{2i}, 0)\,; \\ W^{AB} &= \begin{pmatrix} 0 & -k^{bc}f_{1c}{}^i \\ k^{ac}f_{1c}{}^j & k^{ad}k^{be}f_{2dec}\tilde{\alpha}_0{}^c \end{pmatrix}, \\ V^A &= (f_{1a}{}^i\tilde{\alpha}_0{}^a, 0)\,, \end{aligned} \tag{4.14}$$

for $A = (i, a)$. The corresponding gauge symmetry is given by

$$\begin{aligned} U_j{}^{Ai} &= \left(0, k^{ab}\frac{\partial f_{1b}{}^i}{\partial\tilde{\phi}^j}\right) \\ U_C{}^{AB} &= 0\,, \qquad \text{otherwise}\,; \\ X_j{}^{abc} &= -k^{ad}k^{be}k^{cf}\frac{\partial f_{2def}}{\partial\tilde{\phi}^j}\,, \\ X_D{}^{ABC} &= 0\,, \qquad \text{otherwise}\,, \end{aligned} \tag{4.15}$$

which satisfies the identities (2.3) due to the identities (4.10).

The 3D nonlinear Chern-Simons theory from X to M reduces to a 2D nonlinear gauge theory with two-forms from Σ to E as a sigma model by dimensional reduction.

4.3 3D nonlinear BF theory

We can also construct 3D nonlinear BF theory with nonlinear gauge symmetry as a deformation of the BF gauge theory in three dimensions. This nonlinear gauge theory in three dimensions has the following action [13]:

$$\begin{aligned} S &= S_0 + S_1; \\ S_0 &= \int_X \left(B_{1a} \wedge dA^a - B_{2i} \wedge d\phi^i\right), \\ S_1 &= \int_X (f_{1a}{}^i(\phi)A^aB_{2i} + f_2{}^{ib}(\phi)B_{2i}B_{1b} + \frac{1}{6}f_{3abc}(\phi)A^aA^bA^c + \\ &\quad + \frac{1}{2}f_{4ab}{}^c(\phi)A^aA^bB_{1c} + \frac{1}{2}f_{5a}{}^{bc}(\phi)A^aB_{1b}B_{1c} + \frac{1}{6}f_6{}^{abc}(\phi)B_{1a}B_{1b}B_{1c})\,, \end{aligned} \tag{4.16}$$

where the structure functions $f_1, \cdots, f_6$ satisfy the identities

$$
\begin{aligned}
f_{1e}{}^i f_2{}^{je} + f_2{}^{ie} f_{1e}{}^j &= 0\,, \\
-\frac{\partial f_{1c}{}^i}{\partial \phi^j} f_{1b}{}^j + \frac{\partial f_{1b}{}^i}{\partial \phi^j} f_{1c}{}^j + f_{1e}{}^i f_{4bc}{}^e + f_2{}^{ie} f_{3ebc} &= 0\,, \\
f_{1b}{}^j \frac{\partial f_2{}^{ic}}{\partial \phi^j} - f_2{}^{jc} \frac{\partial f_{1b}{}^i}{\partial \phi^j} + f_{1e}{}^i f_{5b}{}^{ec} - f_2{}^{ie} f_{4eb}{}^c &= 0\,, \\
-f_2{}^{jb} \frac{\partial f_2{}^{ic}}{\partial \phi^j} + f_2{}^{jc} \frac{\partial f_2{}^{ib}}{\partial \phi^j} + f_{1e}{}^i f_6^{ebc} + f_2{}^{ie} f_{5e}{}^{bc} &= 0\,, \\
-f_{1[a}{}^j \frac{\partial f_{4bc]}{}^d}{\partial \phi^j} + f_2{}^{jd} \frac{\partial f_{3abc}}{\partial \phi^j} + f_{4e[a}{}^d f_{4bc]}{}^e + f_{3e[ab} f_{5c]}{}^{de} &= 0\,, \\
-f_{1[a}{}^j \frac{\partial f_{5b]}{}^{cd}}{\partial \phi^j} - f_2{}^{j[c} \frac{\partial f_{4ab}{}^{d]}}{\partial \phi^j} + f_{3eab} f_6{}^{ecd} + f_{4e[a}{}^{[d} f_{5b]}{}^{c]e} + f_{4ab}{}^e f_{5e}{}^{cd} &= 0\,, \\
-f_{1a}{}^j \frac{\partial f_6{}^{bcd}}{\partial \phi^j} + f_2{}^{j[b} \frac{\partial f_{5a}{}^{cd]}}{\partial \phi^j} + f_{4ea}{}^{[b} f_6{}^{cd]e} + f_{5e}{}^{[bc} f_{5a}{}^{d]e} &= 0\,, \\
-f_2{}^{j[a} \frac{\partial f_6{}^{bcd]}}{\partial \phi^j} + f_6{}^{e[ab} f_{5e}{}^{cd]} &= 0\,, \\
-f_{1[a}{}^j \frac{\partial f_{3bcd]}}{\partial \phi^j} + f_{4[ab}{}^e f_{3cd]e} &= 0\,. \qquad (4.17)
\end{aligned}
$$

Note that the 3D theory in subsection 4.1 is an example of this action.

As is the case in the previous subsection, if we take a local basis, eq. (4.17) is equivalent to the relations of structure functions of a Courant algebroid on a vector bundle $E \oplus E^*$ over M: Symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ is defined from the natural pairing of E and E^*. That is, $\langle e_a\,, e_b \rangle = \langle e^a\,, e^b \rangle = 0$ and $\langle e_a\,, e^b \rangle = \delta_a{}^b$ if $\{e_a\}$ is a basis of sections of E and $\{e^a\}$ is that of E^*. The bilinear form $\circ$ and the anchor ρ are represented as follows:

$$
\begin{aligned}
e^a \circ e^b &= -f_{5c}{}^{ab}(\phi) e^c - f_6{}^{abc}(\phi) e_c\,, \\
e^a \circ e_b &= -f_{4bc}{}^a(\phi) e^c + f_{5b}{}^{ac}(\phi) e_c\,, \\
e_a \circ e_b &= -f_{3abc}(\phi) e^c - f_{4ab}{}^c(\phi) e_c\,, \\
\rho(e^a) &= -f_2{}^{ia}(\phi) \frac{\partial}{\partial \phi^i}\,, \\
\rho(e_a) &= -f_{1a}{}^i(\phi) \frac{\partial}{\partial \phi^i}\,. \qquad (4.18)
\end{aligned}
$$

We again consider dimensional reduction of the theory from the three-dimensional manifold $X = \Sigma \times S^1$ to the two-dimensional manifold Σ:

$$
\begin{aligned}
\phi^i &= \tilde{\phi}^i\,, \\
A^a &= \tilde{A}_1{}^a + \tilde{\alpha}_0{}^a d\rho\,, \\
B_{1a} &= \tilde{B}_{1a} + \tilde{\beta}_{0a} d\rho\,, \\
B_{2i} &= \tilde{B}_{2i} + \tilde{\beta}_{1i} d\rho\,, \qquad (4.19)
\end{aligned}
$$

where $\tilde{\phi}^i$ is a reduction of ϕ^i, $\tilde{A}_1{}^a$ and $\tilde{B}_{1a}$ are 1-forms, $\tilde{\alpha}_0{}^a$ and $\tilde{\beta}_{0a}$ are 0-forms, $\tilde{B}_{2i}$ is a 2-form, and $\tilde{\beta}_{1i}$ is a 1-form in two dimensions.

Then the action (4.16) is reduced to the following action:

$$
\begin{aligned}
S = & \int_X \left(\tilde{B}_{1a} \wedge d\tilde{\alpha}_0{}^a + \tilde{A}_1{}^a \wedge d\tilde{\beta}_{0a} + \tilde{\beta}_{1i} \wedge d\tilde{\phi}^i \right) d\rho + d\left(\tilde{A}_1{}^a \tilde{\beta}_{0a} d\rho \right) + \\
& + \Bigg(f_{1a}{}^i (\tilde{A}_1{}^a \tilde{\beta}_{1i} + \tilde{\alpha}_0{}^a \tilde{B}_{2i}) + f_2{}^{ib} (\tilde{B}_{2i} \tilde{\beta}_{0b} - \tilde{\beta}_{1i} \tilde{B}_{1b}) + \frac{1}{2} f_{3abc} \tilde{A}_1{}^a \tilde{A}_1{}^b \tilde{\alpha}_0{}^c + \\
& \quad + \frac{1}{2} f_{4ab}{}^c (\tilde{A}_1{}^a \tilde{A}_1{}^b \tilde{\beta}_{0c} - 2\tilde{A}_1{}^a \tilde{B}_{1c} \tilde{\alpha}_0{}^b) + \frac{1}{2} f_{5a}{}^{bc} (2\tilde{A}_1{}^a \tilde{B}_{1b} \tilde{\beta}_{0c} + \tilde{\alpha}_0{}^a \tilde{B}_{1b} \tilde{B}_{1c}) + \\
& \quad + \frac{1}{2} f_6^{abc} \tilde{B}_{1a} \tilde{B}_{1b} \tilde{\beta}_{0c} \Bigg) d\rho \\
= & \int_\Sigma \tilde{B}_{1a} \wedge d\tilde{\alpha}_0{}^a + \tilde{A}_1{}^a \wedge d\tilde{\beta}_{0a} + \tilde{\beta}_{1i} \wedge d\tilde{\phi}^i + f_{1a}{}^i \tilde{A}_1{}^a \tilde{\beta}_{1i} + f_2{}^{ib} \tilde{B}_{1b} \tilde{\beta}_{1i} + \\
& + \frac{1}{2} (f_{3abc} \tilde{\alpha}_0{}^c + f_{4ab}{}^c \tilde{\beta}_{0c}) \tilde{A}_1{}^a \tilde{A}_1{}^b + (-f_{4ab}{}^c \tilde{\alpha}_0{}^b + f_{5a}{}^{cb} \tilde{\beta}_{0b}) \tilde{A}_1{}^a \tilde{B}_{1c} + \\
& + \frac{1}{2} (f_{5a}{}^{bc} \tilde{\alpha}_0{}^a + f_6^{abc} \tilde{\beta}_{0a}) \tilde{B}_{1b} \tilde{B}_{1c} + (f_{1b}{}^i \tilde{\alpha}_0{}^b + f_2{}^{ia} \tilde{\beta}_{0a}) \tilde{B}_{2i} \, ,
\end{aligned} \tag{4.20}
$$

which can be also obtained through the action (2.1) by letting

$$
\begin{aligned}
h_A &= (\tilde{\beta}_{1i}, \tilde{A}_1{}^a, \tilde{B}_{1b}) \, , \\
\Phi^A &= (\tilde{\phi}^i, \tilde{\beta}_{0a}, \tilde{\alpha}_0{}^b) \, , \\
\tilde{B}_{2A} &= (\tilde{B}_{2i}, 0, 0) \, ; \\
W^{AB} &= \begin{pmatrix} 0 & -f_{1c}{}^i & -f_2{}^{id} \\ f_{1a}{}^j & f_{3ace} \tilde{\alpha}_0{}^e + f_{4ac}{}^e \tilde{\beta}_{0e} & -f_{4ae}{}^d \tilde{\alpha}_0{}^e + f_{5a}{}^{de} \tilde{\beta}_{0e} \\ f_2{}^{jb} & f_{4be}{}^c \tilde{\alpha}_0{}^e - f_{5b}{}^{ce} \tilde{\beta}_{0e} & f_{5e}{}^{bd} \tilde{\alpha}_0{}^e + f_6^{bde} \tilde{\beta}_{0e} \end{pmatrix} , \\
V^A &= (f_{1b}{}^i \tilde{\alpha}_0{}^b + f_2{}^{ia} \tilde{\beta}_{0a}, 0, 0) \, ,
\end{aligned} \tag{4.21}
$$

for ${}^A = ({}^i, {}_a, {}^b)$. The corresponding gauge symmetry is given by

$$
\begin{aligned}
U_j{}^{Ai} &= \left(0, \frac{\partial f_{1a}{}^i}{\partial \tilde{\phi}^j}, \frac{\partial f_2{}^{ib}}{\partial \tilde{\phi}^j} \right) \\
U_C{}^{AB} &= 0 \, , \qquad \text{otherwise} \, ; \\
X_{jabc} &= -\frac{\partial f_{3abc}}{\partial \tilde{\phi}^j} \, , \\
X_{jab}{}^c &= -\frac{\partial f_{4ab}{}^c}{\partial \tilde{\phi}^j} \, , \\
X_{ja}{}^{bc} &= -\frac{\partial f_{5a}{}^{bc}}{\partial \tilde{\phi}^j} \, , \\
X_j{}^{abc} &= -\frac{\partial f_6{}^{abc}}{\partial \tilde{\phi}^j} \, , \\
X_D{}^{ABC} &= 0 \, , \qquad \text{otherwise} \, ,
\end{aligned} \tag{4.22}
$$

where complete antisymmetrization for $X_j{}^{ABC}$ with respect to the indices ABC should be understood. This again satisfies the identities (2.3) due to the identities (4.17).

The 3D nonlinear BF theory from X to M reduces to a 2D nonlinear gauge theory with two-forms from Σ to $E \oplus E^*$ as a sigma model by dimensional reduction.

5. Conclusion

We have provided the 2D nonlinear gauge theory with two-forms, which is obtained as the consistent deformation of 2D topological BF gauge theory.

Dimensional reduction of 3D nonlinear gauge theory based on Courant algebroid such as nonlinear Chern-Simons theory and 3D nonlinear BF theory yields such 2D nonlinear gauge theory with two-forms. If it is reducible as is considered at the end of section 2, we obtain the Poisson sigma model based on Lie algebroid with a reduced target space. Namely, the reduction of the base space is accompanied by that of the target space structure with the Courant algebroid reduced to the Lie algebroid.[4]

We have analyzed the algebroid defined by eq. (2.3) in terms of the Batalin-Vilkovisky algebra in section 3. Further analyses of algebraic and geometric structures of this type of theories would shed more light on relations between topological field theories and algebroids, including a Lie algebroid and a Courant algebroid.

The web of topological gauge field theories of the Schwarz type may be connected by deformations and reductions, as is exemplified in the cases of two- and three-dimensional nonlinear gauge theories in this paper. It can be also deformed to nontopological theories [18] and constitutes an intriguing arena in the space of field theories from a deformation theory perspective.

A. Definitions

The Batalin-Vilkovisky antibracket [15] for functions $F(\varphi,\varphi^+)$ and $G(\varphi,\varphi^+)$ of the fields and antifields on the base space X is defined by

$$(F,\, G)_{\mathrm{AB}} \equiv \frac{F\overleftarrow{\partial}}{\partial\varphi}\frac{\overrightarrow{\partial} G}{\partial\varphi^+} - (-1)^{(n+1)\deg\varphi}\frac{F\overleftarrow{\partial}}{\partial\varphi^+}\frac{\overrightarrow{\partial} G}{\partial\varphi}\,, \tag{A.1}$$

where $n = \dim X$ and $\overleftarrow{\partial}/\partial\varphi$ and $\overrightarrow{\partial}/\partial\varphi$ are the right and left differentiations with respect to φ, respectively, which satisfy

$$\frac{\overrightarrow{\partial} F}{\partial\varphi} = (-1)^{(\mathrm{gh}F+\mathrm{gh}\varphi)\mathrm{gh}\varphi+(\deg F+\deg\varphi)\deg\varphi}\frac{F\overleftarrow{\partial}}{\partial\varphi}\,. \tag{A.2}$$

$\overleftarrow{\partial}/\partial\varphi^+$ and $\overrightarrow{\partial}/\partial\varphi^+$ have similar definitions. When F and G are functionals of the fields φ and antifields φ^+, the antibracket is given by

$$(F,\, G)_{\mathrm{AB}} \equiv \int_X \left(\frac{F\overleftarrow{\partial}}{\partial\varphi}\frac{\overrightarrow{\partial} G}{\partial\varphi^+} - (-1)^{(n+1)\deg\varphi}\frac{F\overleftarrow{\partial}}{\partial\varphi^+}\frac{\overrightarrow{\partial} G}{\partial\varphi}\right). \tag{A.3}$$

The antibracket satisfies the following identities:

$$\begin{aligned}
&(F,\, G)_{\mathrm{AB}} = -(-1)^{(\deg F+n)(\deg G+n)+(\mathrm{gh}F+1)(\mathrm{gh}G+1)}(G,\, F)_{\mathrm{AB}}\,,\\
&(F,\, GH)_{\mathrm{AB}} = (F,\, G)_{\mathrm{AB}}H + (-1)^{(\deg F+n)\deg G+(\mathrm{gh}F+1)\mathrm{gh}G}G(F,\, H)_{\mathrm{AB}}\,,\\
&(FG,\, H)_{\mathrm{AB}} = F(G,\, H)_{\mathrm{AB}} + (-1)^{\deg G(\deg H+n)+\mathrm{gh}G(\mathrm{gh}H+1)}(F,\, H)_{\mathrm{AB}}G\,,\\
&(-1)^{(\deg F+n)(\deg H+n)+(\mathrm{gh}F+1)(\mathrm{gh}H+1)}(F,\, (G,\, H)_{\mathrm{AB}})_{\mathrm{AB}} + \text{cyclic permutations} = 0\,.
\end{aligned} \tag{A.4}$$

[4]In specific cases, higher-dimensional theory itself can be directly based on the Lie algebroid [4, 10] prior to dimensional reduction (see the first case study in the previous section).

We also note that

$$\begin{aligned} FG &= (-1)^{\mathrm{gh}F\mathrm{gh}G+\deg F\deg G}GF\,, \\ d(FG) &= dFG+(-1)^{\deg F}FdG\,. \end{aligned} \tag{A.5}$$

In order to simplify cumbersome sign factors, we introduce super product, super antibracket, and super differentiation [16].[5]

Let us define the *super product* by

$$F\cdot G\equiv(-1)^{\mathrm{gh}F\deg G}FG\,. \tag{A.6}$$

We obtain the following identities from eq. (A.5):

$$\begin{aligned} F\cdot G &= (-1)^{|F||G|}G\cdot F\,, \\ d(F\cdot G) &= dF\cdot G+(-1)^{|F|}F\cdot dG\,, \end{aligned} \tag{A.7}$$

where $|F|\equiv\mathrm{gh}F+\deg F$ denotes the total degree of F.

The *super antibracket* is defined by

$$(F,\,G)\equiv(-1)^{(\mathrm{gh}F+1)(\deg G+n)}(-1)^{\mathrm{gh}\varphi(\deg\varphi+n)+n}(F,\,G)_{\mathrm{AB}}\,. \tag{A.8}$$

Then the following identities are obtained from eq. (A.4):

$$\begin{aligned} &(F,\,G)=-(-1)^{(|F|+n+1)(|G|+n+1)}(G,\,F)\,, \\ &(F,\,G\cdot H)=(F,\,G)\cdot H+(-1)^{(|F|+n+1)|G|}G\cdot(F,\,H)\,, \\ &(F\cdot G,\,H)=F\cdot(G,\,H)+(-1)^{|G|(|H|+n+1)}(F,\,H)\cdot G\,, \\ &(-1)^{(|F|+n+1)(|H|+n+1)}(F,\,(G,\,H))+\text{cyclic permutations}=0\,. \end{aligned} \tag{A.9}$$

That is, the super antibracket provides a graded Poisson bracket on *superfields.*

We further define the *super differentiation* by

$$\begin{aligned} \frac{\overrightarrow{\partial}}{\partial\varphi}\cdot F &\equiv (-1)^{\mathrm{gh}\varphi\deg F}\frac{\overrightarrow{\partial}F}{\partial\varphi}\,, \\ F\cdot\frac{\overleftarrow{\partial}}{\partial\varphi} &\equiv (-1)^{\mathrm{gh}F\deg\varphi}\frac{F\overleftarrow{\partial}}{\partial\varphi}\,. \end{aligned} \tag{A.10}$$

We can define the *super differentiation* with respect to φ^+ in a similar manner. Owing to eq. (A.2), we obtain

$$\frac{\overrightarrow{\partial}}{\partial\varphi}\cdot F=(-1)^{(|F|+|\varphi|)|\varphi|}F\cdot\frac{\overleftarrow{\partial}}{\partial\varphi}\,. \tag{A.11}$$

[5]The super product is called the dot product in ref. [11, 12, 13, 16].

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